Fermions in conformally invariant geometrodynamics

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Abstract

Dynamic equations that are the simplest conformally invariant generalization of Einstein equations with cosmological term are considered. Dimensions and Weyl weights of the additional geometrical fields (the vector and the antisymmetric tensor) appearing in the scheme are such, that they admit an unexpected interpretation. It is proved that the fields can be interpreted as observed, generated by bispinor degrees of freedom. The vacuum polarization density matrix leads to different probabilities of different helicity particle generation.

This paper leans on ref. [1] which proves the below statements and gives needed references.

The simplest conformally invariant generalization of Einstein equations with a nonzero lambda term are the following equations:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -2A_\alpha A_\beta - g_{\alpha\beta} A^2 - 2g_{\alpha\beta} A^\nu{}_{;\nu} + A_{\alpha;\beta} + A_{\beta;\alpha} + \lambda g_{\alpha\beta} \quad . \tag{1}$$

Equations (1) retain their form in conformal transformations of field quantities:

$$\left.\begin{array}{l} g_{\alpha\beta} \to g'_{\alpha\beta} = g_{\alpha\beta} \cdot exp\left[2\phi(x)\right], \\ A_\alpha \to A'_\alpha = A_\alpha - \frac{\partial\phi(x)}{\partial x^\alpha}, \\ \lambda \to \lambda' = \lambda \cdot exp\left[-2\phi(x)\right], \end{array}\right\} \tag{2}$$

where $\phi(x)$ is an arbitrary function of coordinates.

Equations (1) can be considered as a extension of Einstein equations to Weyl space. In the Weyl space each geometrical quantity is assigned the characteristic, such as Weyl weight n, which is an integer. If a quantity has weight n, then in transformations (2) it is multiplied by $exp\left[n \cdot \phi(x)\right]$.

From the above-said and from (2) it follows that the weight of the metric tensor is $n = -2$ and that of the lambda term is $n = +2$. Vector J^α and

antisymmetric tensor $H^{\alpha\beta}$, which will play a specific role in what follows, are:

$$J^{\alpha} \equiv \frac{1}{\lambda^{3/2}} g^{\alpha\beta} \left(\lambda_{;\beta} - 2A_{\beta}\lambda\right), \tag{3}$$

$$H^{\alpha\beta} \equiv \frac{1}{\lambda} g^{\alpha\mu} g^{\beta\nu} \left(A_{\nu;\mu} - A_{\mu;\nu}\right). \tag{4}$$

From equations (1) it follows that

$$\left(\lambda H_{\alpha}{}^{\nu}\right)_{;\nu} = \lambda^{3/2} J_{\alpha}, \tag{5}$$

$$\left(\lambda^{3/2} J^{\alpha}\right)_{;\alpha} = 0. \tag{6}$$

For the following consideration, definition of bispinor matrix Z will be needed. Like in Riemannian space, in Weyl space this matrix is introduced through world and local Dirac matrices γ_{α}, γ_k with relation

$$\gamma_{\alpha}(x) = H_{\alpha}^{k}(x) \cdot Z^{-1}(x)\, \gamma_k Z(x).$$

Here $H_{\alpha}^{k}(x)$ is a system of reference vectors determined by relation $g_{\alpha\beta} = H_{\alpha}^{m} H_{\beta}^{n} g_{mn}$.

Particular role of (3), (4) is that it is only they that can be used to construct Hermitean matrix

$$M = J^{k} \cdot \left(\gamma_k D^{-1}\right) + H^{mn} \cdot \left(S_{mn} D^{-1}\right), \tag{7}$$

all parts of which are dimensionless and have zero Weyl weight.

The tensor system which can be mapped onto bispinors in the 4-dimensional Weyl space is composed of a scalar, a vector, an antisymmetric tensor, a pseudo-vector, and a pseudo-scalar. However, in the case under discussion, among the system only vector (3) and tensor (4) are suitable for the purposes of the mapping onto bispinors.

Matrix (7) is Hermitean. If it is, in addition, positive, then find H - arithmetic root of it.

$$M(x) = H(x)\, H^{+}(x). \tag{8}$$

The square root extraction operation with the requirement of the hermiticity and the root positivity is unique, if matrix M is positive. Thus, equation (8) is an algebraic equation for finding H by Cauchy data for equation (1). Recall that for equations (1) the Cauchy problem is well-posed without any connections to the Cauchy data on the initial hypersurface.

Write the polar decomposition for the bispinor matrix as

$$Z = HU^{-1}. \tag{9}$$

With taking into account (11) we obtain

$$M\left(x\right) = H\left(x\right)H^{+}\left(x\right) = Z\left(x\right)Z^{+}\left(x\right). \tag{10}$$

Vector (3) and anti-symmetric tensor (4) are expressed in terms of the bispinor matrix as follows:

$$J^{\alpha} = \tfrac{1}{4}Sp\left[Z^{+}C\gamma_5\gamma^{\alpha}Z\right], \quad H^{\alpha\beta} = -\tfrac{1}{8}Sp\left[Z^{+}CS^{\alpha\beta}Z\right]. \tag{11}$$

Thus, in the scheme under consideration the Hermitean multiplier $H\left(x\right)$ in (8) is determined by equations (1). As for the unitary multiplier $U^{-1}\left(x\right)$, the simplest conformally invariant equation for it is

$$\left(HD\gamma^{\alpha}H\right)\left(\left(\nabla_{\alpha}U^{-1}\right)U\right) = -\tfrac{1}{2}\left(\nabla_{\alpha}\left(HD\gamma^{\alpha}H\right)\right) - \tfrac{3}{4}\left(HD\gamma^{\alpha}H\right)\cdot\left(ln\lambda\right)_{;\alpha}; \tag{12}$$

Introduce the following notation for the anti-Hermitean matrix vector $iZ^{+}D\gamma^{\alpha}Z$:

$$\Upsilon^{\alpha} \equiv iZ^{+}D\gamma^{\alpha}Z. \tag{13}$$

With using notation (13), equation (12) assumes the following compact form:

$$\left(\nabla_{\alpha}\left(\lambda^{3/2}\cdot\Upsilon^{\alpha}\right)\right) = 0. \tag{14}$$

For matrix connectivity Γ_{α}, determine the matrix curvature tensor (Young-Mills tensor in conventional terms) in the usual manner:

$$P_{\alpha\beta} \equiv \Gamma_{\beta;\alpha} - \Gamma_{\alpha;\beta} + \Gamma_{\alpha}\Gamma_{\beta} - \Gamma_{\beta}\Gamma_{\alpha}.$$

The only possible form of the conformally invariant Young-Mills equation is

$$\left(\nabla_{\nu}P_{\alpha}{}^{\nu}\right) = Const\cdot\left(\lambda^{3/2}\cdot\Upsilon_{\alpha}\right). \tag{15}$$

Relation (14) ensures consistency of the left-hand and right-hand sides in equation (15).

From the method the matrix M is introduced with it follows that it has the meaning of the polarization density matrix. In our case there are two types of bispinor degrees of freedom. The degrees relating to the Hermitean

multiplier determine the matrix M itself, therefore, the matrix M can not be a polarization matrix with respect to them. As for the degrees relating to the unitary multiplier U^{-1}, the matrix M appears in the dynamic equations for U^{-1}. As U^{-1}-related degrees of freedom are external, "nested" bispinor degrees of freedom, then, when treating their dynamics, the matrix M should be considered as the one determining polarization properties of background vacuum. The quantity M (and thereby the quantities A_α and λ) determine relative probabilities of generation of virtual particles with half-integer spin in different states. Thus, the physical meaning of the quantities A_α and λ is that they give properties of the geometrodynamical vacuum, which is background vacuum for other degrees of freedom (that is for the bispinor degrees determined by the unitary multiplier U and for the Young-Mills fields).

Presume that the polarization matrix M is an operator in the space of half-integer spin particle states. That is make an assumption, which is typically made in quantization of the classic field theory apparatus. Then the case of $M = E$ will describe the situation, where virtual particles with half-integer spin can be generated with the same probability in any of possible states.

The question arises: How should matrix M, whose eigenvalues are not equal to one, be treated? A possible treatment is that in the considered space the probabilities of generation of virtual bispinor particles in one state are somewhat higher than those in another state. In the theory of matrix spaces bispinors are separated from the bispinor matrix through multiplication of Z by projector on the left. In the general case the projector can be produced with using matrix $i\gamma_5$ and/or matrix of mapping from group U_4. Among these two matrices the state partition into $2 + 2$ can be achieved only with the $i\gamma_5$ separating the states with the left and right helicity. So in the scheme under discussion the virtual bispinor particles of different helicity have different probabilities of generation.

It is not impossible that the above mechanism leads to appearance of baryon asymmetry in the Universe. A clue to the puzzle A.D. Sakharov was concerned about - What is the reason for the baryon asymmetry in the Universe? - may be along this path.

The above considerations are no more than hypotheses and require further studies. Quite reliable seems the relationship found in this paper between the tensor field dynamics and the bispinor degree of freedom dynamics. Of quite general nature is also the division of the bispinor degrees of freedom into two parts: (1) the part contained in H (Hermitean multiplier in the bispinor

matrix factorization) and rigidly connected to the background space-time vacuum; (2) the part contained in U^{-1} (unitary multiplier in the bispinor matrix factorization) and having (along with the matrix connectivity) the meaning of a field "nested" in a given geometrodynamical background vacuum.

As far as we know, the analysis method used is novel and may be useful for interpretation not only of the scheme based on equation (1), but also of other theories aimed at a unified description of fundamental interactions.

References

[1] M.V. Gorbatenko. *Voprosy Atomnoi Nauki i Tekhniki.* Seriya: Teor. i Prikl. Fizika. **3**, 28-40 (2001) [In Russian].